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On the Vanishing Boundary Terms of Noether's Conservation Laws

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Abstract. In the process of calculating Noether's conservation laws as in [1], two sets of integration by parts are performed. Here it is shown why the boundary terms of the first set of integration by parts vanish.

1. Introduction

In recent work [1], it was shown that for variational problems that are invariant under some symmetry group, Noether's conservation laws could be written in terms of a moving frame and vectors of invariants. These conservation laws come, roughly speaking, from a careful collection of the boundary terms obtained during the calculation of the first variation of $\mathscr{L}[\mathbf{u}] = \int L[\mathbf{u}]\mathrm{d}\mathbf{x}$, where $L[\mathbf{u}]$ is a smooth function of the independent variables $\mathbf{x}$, the dependent variables $\mathbf{u}$, and finitely many derivatives of the u^α. The calculation of the first variation involves two sets of integration by parts. One can notice that Noether's conservation laws have no terms involving the boundary terms from the first set of integration by parts. In this paper, it will be shown why these boundary terms disappear.

This has a great implication; the boundary terms from the first set of integration by parts can be immediately discarded, simplifying the computation of Noether's conservation laws.

Since this article serves merely as a complement to [1], only concepts not mentioned there will be expounded.

In Section 2, notions on invariant differentiation will be introduced. Section 3 will consist of the result on the vanishing boundary terms of Noether's conservation laws.

2. Invariant Differentiation

The moving frames used here are the ones as reformulated by Fels and Olver [2,3], but from the differential algebra point of view. Since this article is a complement to [1], the notation presented here will be the same.

Recall that

$$\frac{\partial}{\partial x_i} u^\alpha_K = u^\alpha_{Ki},$$

although the same does not hold, in general, for their invariant counterparts,

$$\mathcal{D}_i I^\alpha_K \neq I^\alpha_{Ki}.$$

Thus, differentiation and invariantization do not commute. This leads to the following definition.

Definition 2.1. *Invariant differentiation of the J_i and I^α_K are defined, respectively, as*

$$\mathcal{D}_j J_i = \delta_{ij} + N_{ij}, \qquad \mathcal{D}_j I^\alpha_K = I^\alpha_{Kj} + M^\alpha_{Kj},$$

where δ_{ij} is the Kronecker delta, and N_{ij} and M^α_{Kj} are the correction terms*.*

Let G be a group parametrized by $a_1, ..., a_r$, where $r = \dim(G)$, in a neighbourhood of the identity element. The infinitesimals of the prolonged group action with respect to these parameters are

$$\xi^j_i = \left.\frac{\partial \widetilde{x_j}}{\partial a_i}\right|_{g=e}, \qquad \phi^\alpha_{K,i} = \left.\frac{\partial \widetilde{u^\alpha_K}}{\partial a_i}\right|_{g=e}.$$

Since ξ^j_i and $\phi^\alpha_{K,i}$ are functions of the x_i, for $i = 1, ..., p$, u^α, for $\alpha = 1, ..., q$, and u^α_K, we can define

$$\xi^j_i(I) = \xi^j_i(J, I^\beta)$$

and

$$\phi^\alpha_{K,i}(I) = \phi^\alpha_{K,i}(J, I^\beta, I^\beta_L),$$

where the arguments have been invariantized.

The following theorem provides formulae for the correction terms N_{ij} and M^α_{Kj}

Theorem 2.2. *For a left action on the base space and a right moving frame, the $p \times r$* correction matrix K*, which provides the correction terms, is given by*

$$\mathsf{K}_{j\ell} = \left.\widetilde{D_j}\rho_\ell(\widetilde{z})\right|_{g=\rho(z)} = \left((T_e R_\rho)^{-1})\mathcal{D}_j \rho\right)_\ell,$$

where $\rho = (\rho_1, ..., \rho_r)^T$ is in parameter form and $R_\rho : G \to G$ is right multiplication by ρ. The formulae for the correction terms are

$$N_{ij} = \sum_{\ell=1}^{r} \mathsf{K}_{j\ell}\xi^i_\ell(I), \qquad M^\alpha_{Kj} = \sum_{\ell=1}^{r} \mathsf{K}_{j\ell}\phi^\alpha_{K,\ell}(I),$$

where ℓ is the index for the group parameters and $r = \dim(G)$.

The proof of this theorem can be found in page 134 of [4].

The correction matrix K can be computed without explicit knowledge of the frame; it requires only information on the normalization equations and the infinitesimals, as shown in the following theorem.

Theorem 2.3. *Suppose $\{\psi_i(z) = 0, i = 1, ..., r\}$ is the set of normalization equations. Let the n variables occuring in the normalization equations be $\zeta_1, ..., \zeta_n$, where m of these are independent variables, and the remaining $n - m$, are dependent variables and*

their derivatives. Set J *to be the* $n \times r$ *transpose of the Jacobian matrix of the left-hand sides of the normalization equations* $\psi_1, ..., \psi_r$*, with invariantized arguments,*

$$\mathsf{J}_{ij} = \frac{\partial \psi_j(I)}{\partial I(\zeta_j)}.$$

Define T *to be the invariant* $p \times n$ *total derivative matrix*

$$\mathsf{T}_{ij} = I\left(\frac{D}{Dx_i}\zeta_j\right).$$

Moreover, let Φ *denote the matrix of infinitesimals with invariantized arguments* ‡,

$$\Phi_{ij} = \left(\left.\frac{\partial (g \cdot \zeta_j)}{\partial a_i}\right|_{g=e}\right)(I).$$

Then the correction matrix K *providing the correction terms in the process of invariant differentiation in Theorem 2.2, can also be given by*

$$\mathsf{K} = -\mathsf{T}\mathsf{J}(\Phi\mathsf{J})^{-1}.$$

We will skip the proof of this theorem, as it can be found in page 136 of [4].

Example 2.4. *Consider the* $SL(2)$ *action on* $(s, t, x(s,t), u(s,t))$ *space*

$$\widetilde{s} = s, \qquad \widetilde{t} = t, \qquad \widetilde{x} = \frac{ax+b}{cx+d}, \qquad \widetilde{u} = 6c(cx+d) + (cx+d)^2 u,$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1.$$

Taking the normalization equations to be $\widetilde{x} = 0$, $\widetilde{x_s} = 1$, *and* $\widetilde{u} = 0$, *then the* ψ_i *are* $\psi_1(x, x_s, u) = x$, $\psi_2(x, x_s, u) = x_s - 1$ *and* $\psi_3(x, x_s, u) = u$. *So the arguments of the* ψ_i *are* x, x_s *and* u, *and the invariantized normalization equations are* $I^x = 0$, $I^x_1 = 1$ *and* $I^u = 0$.

The following table presents the infinitesimals of the prolonged group action.

	s	t	x	u	x_s	$\cdots$
a	0	0	$2x$	$-2u$	$2x_s$	$\cdots$
b	0	0	1	0	0	$\cdots$
c	0	0	$-x^2$	$6+2xu$	$-2xx_s$	$\cdots$

Selecting the appropriate columns of the table of infinitesimals, we obtain

$$\Phi = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} x & x_s & u \end{array} \\ \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix} \end{array}.$$

‡ Note that both Φ and $\Omega^\alpha(I)$, for $\alpha = 1, ..., q$, are matrices of invariantized infinitesimals, but they are not exactly the same matrices.

The transpose of the Jacobian matrix invariantized, in this case, is equal the identity matrix,

$$\mathsf{J} = \begin{array}{c c} & \begin{array}{ccc} \psi_1 & \psi_2 & \psi_3 \end{array} \\ \begin{array}{c} I^x \\ I^x_1 \\ I^u \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array},$$

and the invariant total derivative matrix is

$$\mathsf{T} = \begin{array}{c c} & \begin{array}{ccc} x & x_s & u \end{array} \\ \begin{array}{c} s \\ t \end{array} & \begin{pmatrix} 1 & I^x_{11} & I^u_1 \\ I^x_2 & I^x_{12} & I^u_2 \end{pmatrix} \end{array}.$$

Hence,

$$\mathsf{K} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} s \\ t \end{array} & \begin{pmatrix} -\frac{1}{2}I^x_{11} & -1 & -\frac{1}{6}I^u_1 \\ -\frac{1}{2}I^x_{12} & -I^x_2 & -\frac{1}{6}I^u_2 \end{pmatrix} \end{array}.$$

Now we possess all the tools needed to prove that no terms in Noether's conservation laws involve the boundary terms coming from the first set of integration by parts.

3. Vanishing Boundary Terms

In [1] we present the result on the structure of Noether's conservation laws. This result shows that Noether's conservation laws can be written as

$$\sum_{i=1}^{p} \mathcal{D}_i \left(\mathcal{A}d(\rho(z))^{-1}\boldsymbol{v}_i(I)\right) = 0,$$

where $\mathcal{A}d(\rho)^{-1}$ is the Adjoint representation of G evaluated at the frame $\rho(z)^{-1}$ and $\boldsymbol{v}_i(I)$ are the vectors of invariants. In the proof of this result, we initially calculate the first variation of $\mathscr{L}[\mathbf{u}]$, which involves two sets of integration by parts. One can verify that the boundary terms from the first set of integration by parts are of the form

$$\mathcal{D}_t I^\alpha_K,$$

where K is the index of differentiation with respect to x_i, for $i = 1, ..., p$. Then, as will be proved in Corollary 3.1, the conflation of t with the group parameters leads to the disappearance of the boundary terms from the first set of integration by parts.

Corollary 3.1. *Let $\{I^\alpha_K\}$ be the set of generating invariants of the prolonged group action $G \times M \to M$, where K is the index of differentiation with respect to x_i, for $i = 1, ..., p$. Furthermore, let t be a dummy invariant independent variable and $(a_1, ..., a_r)$ the group parameters of G. Letting t be a group parameter and setting $t = a_j$, for each group parameter, then $\mathcal{D}_t I^\alpha_K$ equals zero.*

Proof. According to Definition 2.1, the derivative of I^α_K is equal to

$$\mathcal{D}_t I^\alpha_K = I^\alpha_{Kt} + M^\alpha_{Kt},$$

which by Theorems 2.2 and 2.3 is equivalent to

$$\mathcal{D}_t I^\alpha_K = \begin{pmatrix} (\mathsf{T}_{t*}) & I^\alpha_{Kt} \end{pmatrix} \begin{pmatrix} -\mathsf{J}(\Phi\mathsf{J})^{-1}\phi^\alpha_K \\ 1 \end{pmatrix}, \tag{1}$$

where (T_{t*}) corresponds to row t of T. From the proof of Theorem 3 of [1], we know that after we conflate t with a group parameter a_j of G,

$$\begin{pmatrix} I^\alpha_t & I^\alpha_{j_1 t} & I^\alpha_{j_1 j_2 t} & \cdots \end{pmatrix} = \left(\mathcal{A}d(\rho)^{-1}_{j*}\right)\Omega^\alpha(I), \tag{2}$$

where $j = 1, ..., r$ and $(\mathcal{A}d(\rho)^{-1}_{j*})$ corresponds to row j of $\mathcal{A}d(\rho)^{-1}$. Since Equation (1) can be rearranged as

$$\mathcal{D}_t I^\alpha_K = \sum_\alpha \begin{pmatrix} I^\alpha_t & I^\alpha_{j_1 t} & \cdots \end{pmatrix} \mathcal{B}^\alpha, \tag{3}$$

we can substitute $\begin{pmatrix} I^\alpha_t & I^\alpha_{j_1 t} & \cdots \end{pmatrix}$ in (3) by (2) for each group parameter and obtain

$$(\mathcal{A}d(\rho)^{-1}_{j*}) \sum_\alpha \Omega^\alpha(I)\mathcal{B}^\alpha, \qquad j = 1, ..., r. \tag{4}$$

Writing the above in matrix form yields

$$\mathcal{A}d(\rho)^{-1} \sum_\alpha \Omega^\alpha(I)\mathcal{B}^\alpha,$$

which is equivalent to

$$\mathcal{A}d(\rho)^{-1} \begin{pmatrix} \Phi & \phi^\alpha_K \end{pmatrix} \begin{pmatrix} -\mathsf{J}(\Phi\mathsf{J})^{-1}\phi^\alpha_K \\ 1 \end{pmatrix} = \mathcal{A}d(\rho)^{-1} \begin{pmatrix} -\phi^\alpha_K + \phi^\alpha_K \end{pmatrix} = \mathbf{0}.$$

□

Example 2.4 (cont.) *Consider the invariant I^x_{11}, then*

$$\mathcal{D}_t I^x_{11} = \begin{pmatrix} I^x_2 & I^x_{12} & I^u_2 & I^x_{112} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} 2I^x_{11} \\ 0 \\ -2 \end{pmatrix} \\ 1 \end{pmatrix}.$$

Conflating t with the group parameters of $SL(2)$ and then writing in matrix form yields

$$\mathcal{A}d(\rho)^{-1} \begin{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix} & \begin{pmatrix} 2I^x_{11} \\ 0 \\ -2 \end{pmatrix} \end{pmatrix} \begin{pmatrix} 0 \\ -I^x_{11} \\ \frac{1}{3} \\ 1 \end{pmatrix} = \mathbf{0},$$

where $\mathcal{A}d(\rho)^{-1}$ is the Adjoint representation of $SL(2)$ as shown in page 27 of [1].

4. Conclusion

Here we demonstrated that the vectors of invariants of Noether's conservation laws do not involve any of the boundary terms from the first set of integration by parts. This simplifies significantly the calculation of Noether's conservation laws, as these boundary terms can be immediately discarded.

References

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